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# Atomicity and Determinism in Boolean Systems

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*Abstract*

The logic of a Boolean system of finite degrees of freedom is shown to be atomic if and only if the system obeys a deterministic theory. This is, therefore, the physical meaning of atomicity. Furthermore, it is proved that nondeterminacy of such a system implies the nonexistence of phase space.

## I

In this note classical mechanics is again considered as a special case of generalised physical theory defined by Kronfli (1970a). The same notation will be employed. We distinguish between a classical theory and a (classical) deterministic one. The former is one in which each pair of observables is compatible. Axiomatically, therefore, a classical system of finite degrees of freedom will necessarily have a logic $\mathscr{L}$ which is a countably generated Boolean $\sigma$-algebra.

We define a *deterministic* system to be a classical one such that there exists a non-empty subset $\mathscr{S}_0$ of the set of its states $\mathscr{S}$ where for each state in $\mathscr{S}_0$ at least one observable has zero variance and furthermore $\mathscr{S}_0$ is an invariant of the dynamical subgroup $\mathscr{D}$ of the group Aut($\mathscr{S}$) of convex automorphisms of $\mathscr{S}$. This apparently weak condition gives the full determinism of classical mechanics.

The meaning of lattice atomicity for the logic of a classical mechanical system has been obscured in the literature. This note shows that atomicity is both a necessary and sufficient condition for a Boolean system of finite degrees of freedom to be deterministic. In other words, the two are synonymous.

Unlike a countably generated $\sigma$-field of subsets of a set, an abstract countably generated Boolean $\sigma$-algebra need not be atomic. (Take, for example, the quotient of all the Borel subsets of the unit interval on the line modulo the sets of Lebesgue measure zero.) In a previous paper (Kronfli, 1970b), atomicity of the (Boolean) logic of a classical system of finite degrees of freedom, was shown to be a sufficient condition for the system to be deterministic. A Boolean system, although classical, need not be deterministic. In this paper it is shown that atomicity is also a necessary condition. It will also be shown that without the condition of determinism, classical phase space need not even exist. The reader may compare the above-mentioned results with the objections to atomicity stated by Birkhoff & von Neumann (1936).

## II

From now on $\mathscr{L}$ is a countably generated Boolean $\sigma$-algebra (the logic of a classical system $S$ of finite degrees of freedom), $\mathscr{S}$ is the set of all probability measures on $\mathscr{L}$ (the states of $S$) and $\mathscr{P}$ the set of all extreme points of the convex set $\mathscr{S}$ (the pure states of $S$). Our first result shows the equivalence of $\mathscr{P} \neq \varnothing$ to the atomicity of $\mathscr{L}$.

*Proposition* 1

*The set $\mathscr{P}$ is not empty if and only if $\mathscr{L}$ is atomic. In this case $\mathscr{P} = \{q_a : a \in \mathscr{A}\}$, where $\mathscr{A}$ is the set of atoms of $\mathscr{L}$ and $q_a$ is the atomic measure concentrated at the atom a.*

*Proof*: Assume $\mathscr{L}$ is atomic. Then clearly for each $a \in \mathscr{A}$, $q_a \in \mathscr{P}$. Thus $\mathscr{P} \neq \varnothing$. That $\{q_a : a \in \mathscr{A}\}$ equals $\mathscr{P}$ follows from Theorem 3 of Kronfli (1970b).

Conversely, assume $\mathscr{P} \neq \varnothing$ and $p \in \mathscr{P}$. First we assert that range$(p) = \{0,1\}$. If this is not so, then there exists $a \in \mathscr{L} \backslash \{\varnothing, 1\}$ such that $0 < p(a) < 1$. Define $p_1, p_2 \in \mathscr{S}$ by

$$p_1(x) = (p(a))^{-1} p(x \wedge a) \qquad (x \in \mathscr{L})$$
$$p_2(x) = (1 - p(a))^{-1} p(x \wedge a')$$

This gives

$$p = p(a).p_1 + (1 - p(a)).p_2$$

with $p_1 \neq p_2$, since $p_1(a) = 1$ and $p_2(a) = 0$. This is a contradiction, since $p$ is an extreme point of the convex set $\mathscr{S}$. Hence, range$(p) = \{0,1\}$. Now let $(a_n) \subset \mathscr{L}$ generate $\mathscr{L}$. Since $p(x)$ is either 0 or 1 for each $x \in \mathscr{L}$, we choose $b_n = a_n$ or $b_n = a_n'$ such that $p(b_n) = 1$ for all $n$. Clearly $(b_n)$ generates $\mathscr{L}$. Put $b = \bigwedge_n b_n$. Since $p(b) = 1$ then $b \neq \varnothing$. Let

$$\mathscr{R} = \{x \in \mathscr{L} : b < x \text{ or } b < x'\}$$

Clearly, $\mathscr{R}$ is a Boolean sub-$\sigma$-algebra of $\mathscr{L}$ containing its generators $(b_n)$ and, therefore, $\mathscr{R} = \mathscr{L}$. But since $b \neq \varnothing$, the above result can not be possible unless $b$ is an atom of $\mathscr{L}$. For, let $x \in \mathscr{L}$ and $x < b$. Since $\mathscr{R} = \mathscr{L}$, then either $b < x$ or $b < x'$. The first case gives $x = b$ and the second $x = \varnothing$. Hence $b$ is an atom.

With the assumption $\mathscr{P} \neq \varnothing$ we proved that $\mathscr{A} \neq \varnothing$. It remains to show that $\mathscr{L}$ is atomic, i.e. each non-zero element of $\mathscr{L}$ dominates at least one atom. Let, therefore $\Omega$, $\Omega_c$, $\mathscr{B}(\Omega)$, $\varphi$ and $\gamma$ be as in Theorems 1 and 2 of Kronfli (1970b). Since $\varphi$ maps $\mathscr{B}(\Omega)$ onto $\mathscr{L}$, then for each non-zero and non-atomic element $a \in \mathscr{L}$ there exists $A \in \mathscr{B}(\Omega)$, which is not a singleton, since $\gamma$ is a bijection on $\Omega_c$ onto $\mathscr{A}$, with $A \cap \Omega_c \neq \varnothing$ and $\varphi(A) = a$. (It is easy to see that this corollary to Theorem 2 holds simply with the assumption $\mathscr{A} \neq \varnothing$.) Let $x \in A \cap \Omega_c$. Then $\gamma(x)$ is an atom and $\gamma(x) < a$. This completes the proof. ∎

Using the definition of determinism given in the first section, we show that determinism and atomicity of the logic are equivalent. The obvious candidate for $\mathscr{S}_0$ is $\mathscr{P}$.

*Proposition* 2

*The countably generated Boolean $\sigma$-algebra $\mathscr{L}$ is the logic of a deterministic system if and only if $\mathscr{L}$ is atomic.*

*Proof*: Let $p \in \mathscr{S}$ and $u$ be an observable whose variance in $p$ is zero. Let $\tau$ be its expectation value,

$$\int_R t\, p \circ u(dt)$$

which is necessarily finite. Then

$$\int_R (t - \tau)^2\, p \circ u(dt) = 0$$

Hence the function $t \to (t - \tau)^2$ on $R \to R$ is zero $(p \circ u)$ almost everywhere. This is possible only if $p \circ u$ is an atomic measure concentrated at $\tau$. Put $a = u(\{\tau\})$. Then $a$ is either $\varnothing$ or an atom of $\mathscr{L}$. But $p(a) = 1$, and hence $a$ is an atom and $p = q_a \in \mathscr{P}$. Thus $\mathscr{P}$ is not empty and by Proposition 1 $\mathscr{L}$ is atomic. Thus at least $\mathscr{S}_0 \subset \mathscr{P}$. Also, each element of Aut($\mathscr{S}$), and hence of the dynamical group $\mathscr{D}$, is a bijection of $\mathscr{P}$ onto itself. Therefore determinism implies the atomicity of $\mathscr{L}$.

Conversely, atomicity of $\mathscr{L}$ implies that the theory of the system is deterministic as was shown in Kronfli (1970b). ∎

*Corollary*

*Let $\mathscr{L}$ be the logic of a Boolean system S of finite degrees of freedom. Then the phase space of S does not exist if S is non-deterministic.*

*Proof*: Assume $S$ is non-deterministic with non-empty phase space. Since phase space is equipotent to $\mathscr{P}$ (Kronfli, 1970b), then $\mathscr{P}$ is not empty. From Propositions 1 and 2, $S$ is deterministic, which is a contradiction. ∎

## *References*

Birkhoff, G. and von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823.
Kronfli, N. S. (1970a). *International Journal of Theoretical Physics*, Vol. 3, No. 3, p. 199.
Kronfli, N. S. (1970b). *International Journal of Theoretical Physics*, Vol. 3, No. 5, p. 395.